SUPPLEMENTARY MATERIAL

6. NOTATION

Let $N(x;\mu,\Sigma)$ be the normal density at x with mean μ and variance Σ, and $\widetilde{f}_n(s \mid \theta) = N\{s; s(\theta), A(\theta)/a_n^2\}$, the asymptotic distribution of the summary statistic. We define $a_{n,\varepsilon} = a_n$ if $\lim_{n\to\infty} a_n\varepsilon_n < \infty$ and $a_{n,\varepsilon} = \varepsilon_n^{-1}$ otherwise, and $c_\varepsilon = \lim_{n\to\infty} a_n\varepsilon_n$, both of which summarize how ε_n decreases relative to the converging rate, a_n, of S_n in Condition 2 below. Define the standardized random variables $W_n(S_n) = a_nA(\theta)^{-1/2}\{S_n - s(\theta)\}$ and $W_{\rm obs} = a_nA(\theta)^{-1/2}\{s_{\rm obs} - s(\theta_0)\}$ and $\beta_0 = I(\theta_0)^{-1}Ds(\theta_0)^TA(\theta_0)^{-1}$ according to Condition 2 below.

Let $f_{W_n}(w \mid \theta)$ and $\widetilde{f}_{W_n}(w \mid \theta)$ be the density for $W_n(S_n)$ when $S_n \sim f_n(\cdot \mid \theta)$ and $\widetilde{f}_n(\cdot \mid \theta)$ respectively. Let $B_\delta = \{\theta \mid \|\theta-\theta_0\| \le \delta\}$ for $\delta > 0$. Define the initial density truncated in B_δ, i.e. $r_n(\theta)\mathbb{I}_{\theta\in B_\delta}/\int_{B_\delta} r_n(\theta)\,d\theta$, by $r_\delta(\theta)$. Let $t(\theta) = a_{n,\varepsilon}(\theta-\theta_0)$ and $v(s) = \varepsilon_n^{-1}(s - s_{\rm obs})$. For any $A \in \mathcal{B}^p$ where $\mathcal{B}^p$ is the Borel sigma-field on $\mathbb{R}^p$, let $t(A)$ be the set $\{\phi : \phi = t(\theta) \text{ for some } \theta \in A\}$. For a non-negative function $h(x)$, integrable in $\mathbb{R}^l$, denote the normalized function $h(x)/\int_{\mathbb{R}^l} h(x)\,dx$ by $h(x)^{(\rm norm)}$. For a function $h(x)$, denote its gradient by $D_xh(x)$, and for simplicity, omit θ from D_θ. For a sequence x_n, we use the notation $x_n = \Theta(a_n)$ to mean that there exist some constants m and M such that $0 < m <\mid x_n/a_n \mid< M < \infty$.

7. ADDITIONAL CONDITIONS

Condition 7. The kernel satisfies

(i) $\int vK_\varepsilon(v)dv = 0$;

(ii)$\prod_{k=1}^l v_{i_k}K_\varepsilon(v)dv < \infty$ for any coordinates $(v_{i_1},\ldots,v_{i_l})$ of v and $l \le p+6$;

(iii)$K_\varepsilon(v) \propto K_\varepsilon(\|v\|_\Lambda^2)$ where $\|v\|_\Lambda^2 = v^T\Lambda v$ and Λ is a positive-definite matrix, and $K(v)$ is a decreasing function of $\|v\|_\Lambda$; (iv) $K_\varepsilon(v) = O(\exp\{-c_1\|v\|^{\alpha_1}\})$ for some $\alpha_1 > 0$ and $c_1 > 0$ as $\|v\| \to \infty$.

Condition 8. There exists α_n satisfying $\alpha_n/a_n^{2/5} \to \infty$ and a density $r_{max}(w)$ satisfying Condition 7(ii)–(iii) where $K_\varepsilon(v)$ is replaced with $r_{max}(w)$, such that $\sup_{\theta\in B_\delta} \alpha_n \mid f_{W_n}(w \mid \theta) - \widetilde{f}_{W_n}(w \mid \theta) \mid\le c_3r_{max}(w)$ for some positive constant c_3.

Condition 9. The following statements hold:

(i) $r_{max}(w)$ satisfies Condition 7(iv); and

(ii) $\sup_{\theta\in B_\delta^C} \widetilde{f}_{W_n}(w \mid \theta) = O(e^{-c_2\|w\|^{\alpha_2}})$ as $\|w\| \to \infty$ for some positive constants c_2 and α_2, and $A(\theta)$ is bounded in $\mathcal{P}$.

Condition 10. The first two moments, $\int_{\mathbb{R}^d} s\widetilde{f}_n(s \mid \theta)ds$ and $\int_{\mathbb{R}^d} s^Ts\widetilde{f}_n(s \mid \theta)ds$, exist.

8. PROOF FOR THEOREM 2

Let

$$\tilde{Q}(\theta \in A \mid s) = \textstyle\int_A r_\delta(\theta)\widetilde{f}_n(s \mid \theta)\,d\theta/\int_{\mathbb{R}^p} r_\delta(\theta)\widetilde{f}_n(s \mid \theta)\,d\theta.$$

Lemma 2. Assume Condition 3–8. If $\varepsilon_n = O(a_n^{-1})$, for any fixed $\nu \in \mathbb{R}^d$ and small enough δ,

$$\sup_{A\in\mathfrak{B}^p} \Big|\tilde{Q}\{a_n(\theta-\theta_0) \in A \mid s_{\rm obs} + \varepsilon_n\nu\} - \int_A N[t; \beta_0\{A(\theta_0)^{1/2}W_{\rm obs} + c_\varepsilon\nu\}, I(\theta_0)^{-1}]dt\Big| \to 0,$$

in probability as $n \to \infty$, where $\beta_0 = I(\theta_0)^{-1}Ds(\theta_0)^TA(\theta_0)^{-1}$.

Proof of Lemma 2: This result generalizes Lemma A1 in [1]. With Lemma A1 from [1], it is sufficient to show that

$$\sup_{A\in\mathfrak{B}^p} \mid \tilde{Q}\{t(\theta) \in A \mid s_{\rm obs} + \varepsilon_n\nu\} - \tilde{\Pi}\{t(\theta) \in A \mid s_{\rm obs} + \varepsilon_n\nu\} \mid$$

is $o_P(1)$ where $\tilde{\Pi}$ denotes the posterior distribution with prior $\pi_\delta(\theta)$ and likelihood $\tilde{f}_n(s \mid \theta)$. With the transformation $t = t(\theta)$ and $v = v(s)$, the left hand side of the above equation can be written as

$$\sup_{A\in\mathfrak{B}^p} \mid \frac{\int_A r_\delta(\theta_0 + a_n^{-1}t)\widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt}{\int_{\mathbb{R}^p} r_\delta(\theta_0 + a_n^{-1}t)\widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt} - \frac{\int_A \pi_\delta(\theta_0 + a_n^{-1}t)\widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt}{\int_{\mathbb{R}^p} \pi_\delta(\theta_0 + a_n^{-1}t)\widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt} \mid . \quad (8.1)$$

For a function $\tau : \mathbb{R}^p \to \mathbb{R}$, define the following auxiliary functions,

$$\phi_1\{\tau(\theta); n\} = \frac{\int_{t(B_\delta)} |\tau(\theta_0+a_n^{-1}t) - \tau(\theta)|\widetilde{f}_n(s_{\rm obs}+\varepsilon_n\nu|\theta_0+a_n^{-1}t)\,dt}{\int_{t(B_\delta)} \tau(\theta_0+a_n^{-1}t)\widetilde{f}_n(s_{\rm obs}+\varepsilon_n\nu|\theta_0+a_n^{-1}t)\,dt},$$

$$\phi_2\{\tau(\theta); n\} = \frac{\tau(\theta)\int_{t(B_\delta)} \widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt}{\int_{t(B_\delta)} \tau(\theta_0 + a_n^{-1}t)\widetilde{f}_n(s_{\rm obs} + \varepsilon_n\nu \mid \theta_0 + a_n^{-1}t)dt}.$$

Then by adding and subtracting $\phi_2\{\tau_n^{-p}r_\delta(\theta); n\}\phi_2\{\pi(\theta); n\}$ in the absolute sign of (8.1), (8.1) can be bounded by

$$\phi_1\{\tau_n^{-p}r_\delta(\theta); n\} + \phi_1\{\pi(\theta); n\}\phi_2\{\tau_n^{-p}r_\delta(\theta); n\} + \phi_1\{\tau_n^{-p}r_\delta(\theta); n\}\phi_2\{\pi(\theta); n\} + \phi_1\{\pi(\theta); n\}.$$

Consider a class of function $\tau(\theta)$ satisfying the following conditions:

(1) There exists a series $\{k_n\}$, such that $\sup_{\theta\in\mathcal{P}_0} \|k_n^{-1}D\tau(\theta)\| < \infty$ and $k_n = o(a_n)$;

(2) $\tau(\theta_0) > 0$ and $\tau(\theta) \in C^1(B_\delta)$.

By Conditions 3–6, $\tau_n^{-p}r_\delta(\theta)$ and $\pi_\delta(\theta)$ belong to the above class. Then if $\phi_1\{\tau(\theta); n\}$ is $o_p(1)$ and $\phi_2\{\tau(\theta); n\}$ is $O_p(1)$, (8.1) is $o_p(1)$ and the lemma holds.

First, from the properties of $\tau(\theta)$ mentioned above, there exists an open set $\omega \subset B_\delta$ such that $\inf_{\theta\in\omega} \tau(\theta) > c_1$, for a constant $c_1 > 0$. Then for $\phi_2\{\tau(\theta); n\}$, it is bounded by

$$\frac{\tau(\theta)}{c_1 \int_{t(\omega)} \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu \mid \theta_0 + a_n^{-1}t)^{(norm)} dt},$$

where $h(x)^{(norm)}$ represents the normalized version of $h(x)$. From equation (7) in the supplementary material of [2], $\widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu \mid \theta_0 + a_n^{-1}t)$ can be written in the following form,

$$a_n^d \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu \mid \theta_0 + a_n^{-1}t) = \frac{1}{\|B_n(t)\|^{1/2}} N[C_n(t)\{A_n(t)t - b_n\nu - c_2\}; \theta_0, I_d], \quad (8.2)$$

where $A_n(t)$ is a series of $d \times p$ matrix functions, $\{B_n(t)\}$ and $\{C_n(t)\}$ are a series of $d \times d$ matrix functions, b_n converges to a non-negative constant and c_2 is a constant, and the minimum of absolute eigenvalues of $A_n(t)$ and eigenvalues of $B_n(t)$ and $C_n(t)$ are all bounded and away from 0. Then for fixed ν, by continuous mapping, (8.2) is away from zero with probability one. Therefore $\phi_2\{\tau(\theta); n\} = O_P(1)$.

Second, by Taylor expansion, $\tau(\theta_0 + a_n^{-1}t) = \tau(\theta_0) + a_n^{-1} D\tau(\theta_0 + e_t t)t$, where $\|e_t\| \leq a_n^{-1}$. Then $\phi_1\{\tau(\theta); n\}$ is equal to

$$\frac{k_n \phi_2\{\tau(\theta); n\}}{a_n \tau(\theta)} \frac{\int_{t(B_\delta)} |k_n^{-1} D\tau(\theta_0 + e_t t)t| \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu | \theta_0 + a_n^{-1}t)\, dt}{\int_{t(B_\delta)} \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu | \theta_0 + a_n^{-1}t)\, dt} \leq \left(\frac{k_n \phi_2\{\tau(\theta); n\}}{a_n \tau(\theta)} \sup_{\theta \in B_\delta} \|k_n^{-1} D\tau(\theta)\|\right) \times \left(\frac{\int_{t(B_\delta)} \|t\| a_n^d \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu | \theta_0 + a_n^{-1}t) dt}{\int_{t(B_\delta)} a_n^d \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu | \theta_0 + a_n^{-1}t)\, dt}\right), \quad (8.3)$$

where the inequality holds by the triangle inequality. By the expression (8.2) and Lemma 7 in the supplementary material of [2], the right hand side of (8.3) is $O_P(1)$. Then together with $\phi_2\{\tau(\theta); n\} = \Theta_P(1)$, $\phi_1\{\tau(\theta); n\} = o_P(1)$. Therefore the Lemma holds. □

Define the joint density of (θ, s) in Algorithm 1 and its approximation, where the s-likelihood is replaced by its Gaussian limit and $r_n(\theta)$ by its truncation, by $q_\varepsilon(\theta, s)$ and $\tilde{q}_\varepsilon(\theta, s)$. Then

$$q_\varepsilon(\theta, s) = \frac{r_n(\theta) f_n(s|\theta) K_{\varepsilon_n}(s - s_{\text{obs}})}{\int_{\mathbb{R}^p \times \mathbb{R}^d} r_n(\theta) f_n(s|\theta) K_{\varepsilon_n}(s - s_{\text{obs}})\, d\theta ds},$$

$$\tilde{q}_\varepsilon(\theta, s) = \frac{r_\delta(\theta) \widetilde{f}_n(s|\theta) K_{\varepsilon_n}(s - s_{\text{obs}})}{\int_{\mathbb{R}^p \times \mathbb{R}^d} r_\delta(\theta) \widetilde{f}_n(s|\theta) K_{\varepsilon_n}(s - s_{\text{obs}})\, d\theta ds}.$$

Let $\tilde{Q}_\varepsilon(\theta \in A \mid s_{\text{obs}})$ be the approximate confidence distribution function equal to $\int_A \int_{\mathbb{R}^d} \tilde{q}_\varepsilon(\theta, s)\, ds d\theta$. With the transformation $t = t(\theta)$ and $v = v(s)$, let

$$\tilde{q}_{\varepsilon, t\nu}(t, v) = \tau_n^{-p} r_\delta(\theta_0 + a_{n,\varepsilon}^{-1}t) \widetilde{f}_n(s_{\text{obs}} + \varepsilon_n \nu \mid \theta_0 + a_{n,\varepsilon}^{-1}t) K_\varepsilon(\nu)$$

be the transformed and unnormalized $\tilde{q}_\varepsilon(\theta, s)$, and

$$\tilde{q}_{A,tv}(h) = \int_A \int_{\mathbb{R}^d} h(t, v) \tilde{q}_{\varepsilon, t\nu}(t, v)\, dv dt$$

for any function $h(\cdot, \cdot)$ in $\mathbb{R}^p \times \mathbb{R}^d$. Denote the factor of $\tilde{q}_{\varepsilon, t\nu}(t, v)$, $\tau_n^{-p} r_\delta(\theta_0 + a_{n,\varepsilon}^{-1}t)$, by $\gamma_n(t)$. Let $\gamma = \lim_{n\to\infty} \tau_n^{-p} r_\delta(\theta)$ and $\gamma(t) = \lim_{n\to\infty} \tau_n^{-p} r_\delta(\theta_0 + \tau_n^{-1}t)$, the limits of $\gamma_n(t)$ when $a_{n,\varepsilon} = a_n$ and $a_{n,\varepsilon} = \tau_n$ respectively. By Condition 4 and 5, $\gamma(t)$ exists and γ is non-zero with positive probability.

Next several functions of t and v defined in [1, proofs for Section 3.1] and relate to the limit of $\tilde{q}_{\varepsilon, t\nu}(t, v)$ are used, including $g(v; A, B, c)$, $g_n(t, v)$, $G_n(v)$ and $g_n'(t, v)$.

Furthermore, several functions defined by integration as following are used: for any $A \in \mathfrak{B}^p$, let

$$g_{A,r}(h) = \int_{\mathbb{R}^d} \int_{t(A)} h(t, v) \gamma_n(t) g_n(t, v)\, dt dv,$$

$$G_{n,r}(v) = \int_{t(B_\delta)} \gamma_n(t) g_n(t, v)\, dt,$$

$$q_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta, s) r_n(\theta) f_n(s \mid \theta) K_\varepsilon(s - s_{\text{obs}}) \varepsilon_n^{-d}\, ds d\theta,$$

$$\tilde{q}_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta, s) r_\delta(\theta) \widetilde{f}_n(s \mid \theta) K_\varepsilon(s - s_{\text{obs}}) \varepsilon_n^{-d}\, ds d\theta,$$

which generalize those defined in [1, proofs for Section 3.1] for the case $r_n(\theta) = \pi(\theta)$.

Lemma 3. Assume Condition 3–7. If $\varepsilon_n = o(a_n^{-1/2})$, then

(i) $\int_{\mathbb{R}^d} \int_{t(B_\delta)} |\tilde{q}_{\varepsilon, t\nu}(t, \nu) - \gamma_n(t) g_n(t, \nu)|\, dt d\nu = o_p(1)$;
(ii) $g_{B_\delta, r}(1) = \Theta_P(1)$;
(iii) $\tilde{q}_{B_\delta, tv}(t^{k_1} v^{k_2}) / \tilde{q}_{B_\delta, tv}(1) = g_{B_\delta, r}(t^{k_1} v^{k_2}) / g_{B_\delta, r}(1) + O_P(a_{n,\varepsilon}^{-1}) + O_P(a_n^2 \varepsilon_n^4)$ for pairs $(k_1, k_2) = (0,0), (1,0), (1,1), (0,1)$, and $(0,2)$;
(iv) $\tilde{q}_{B_\delta}(1) =$

$$\tau_n^p a_{n,\varepsilon}^{d-p} \left\{\int_{t(B_\delta)} \int_{\mathbb{R}^d} \gamma_n(t) g_n(t, \nu) d\tau d\nu + O_P(a_{n,\varepsilon}^{-1}) + O_P(a_n^2 \varepsilon_n^4)\right\}.$$

Proof of Lemma 3: These results generalize parts of Lemma A2 in [1] (corresponding to items (*i*) and (*ii*) above) and Lemma 5 in [2] (corresponding to items (*iii*) and (*iv*) above).

To prove part (*i*), note that in Lemma A2 of [1] $\gamma_n(t) = \pi(\theta_0 + a_{n,\varepsilon}^{-1}t)$, and (*i*) holds by expanding $\tilde{q}_{\varepsilon, tv}(t, v)$ according to the proof of Lemma 5 of [2]. For $\gamma_n(t) = \tau_n^{-p} r_\delta(\theta_0 + a_{n,\varepsilon}^{-1}t)$ this can be similarly proved by changing the terms involving $\pi(\theta)$ in equations (10) and (11) in the supplements of [2]. Equation (10) is replaced by $\frac{\gamma_n(t)}{|A(\theta + a_{n,\varepsilon}^{-1}t)|^{1/2}} = \frac{\gamma_n(t)}{|A(\theta)|^{1/2}} + a_{n,\varepsilon}^{-1} \gamma_n(t) D \frac{1}{|A(\theta + e_t)|^{1/2}} t$, where $\|e_\tau\| \leq \delta$, and this leads to replacing $\pi(\theta_0) \int_{\tau(B_\delta) \times \mathbb{R}^d} g_n(t, \nu) dt d\nu$ in equation (11) by

$\int_{\tau(B_\delta)\times\mathbb{R}^d}\gamma_n(t)g_n(t,\nu)dtd\nu$. These changes have no effect on the arguments therein since $\sup_{t\in t(B_\delta)}\gamma_n(t)=O_P(1)$ by Condition 4. Therefore (i) holds.

For (ii), By Condition 5 and Lemma A2 of [1], there exists a $\delta'<\delta$ such that $\inf_{t\in t(B_{\delta'})}\gamma_n(t)=\Theta_p(1)$ and $\int_{\mathbb{R}^d}\int_{t(B_{\delta'})}g_n(t,\nu)\,dtdv=\Theta_p(1)$. Then since $g_{B_\delta,r}(1)\geq\inf_{t\in t(B_{\delta'})}\gamma_n(t)\int_{\mathbb{R}^d}\int_{t(B_{\delta'})}g_n(t,\nu)\,dtd\nu$, (ii) holds.

For (iii), if $(k_1,k_2)=(1,0)$ then $\tilde q_{B_\delta,tv}(t)/\tilde q_{B_\delta,tv}(1)$ can be expanded by following the arguments in the proof of Lemma 5 of [2]. For the other pairs of (k_1,k_2), $\tilde q_{B_\delta,tv}(t^{k_1}v^{k_2})/\tilde q_{B_\delta,tv}(1)$, can be expanded similarly as in the proof of Lemma 4 from [1].

For (iv), $\gamma_n(t)$ plays the same role as $\pi(\theta)$ in the proof of Lemma 5 in [2], and the arguments therein can be followed exactly. The term τ_n^p is from the definition of $\gamma_n(t)$ that $r_n(\theta_0+a_{n,\varepsilon}^{-1}t)=\tau_n^p\gamma_n(t)$. □

Recall the definition of the estimator $\theta_\varepsilon=\int\theta dQ_\varepsilon(\theta\mid s_{obs})d\theta$. Define the expectation of θ with distribution $\tilde Q_\varepsilon(\theta\in A\mid s_{\rm obs})$ as $\tilde\theta_\varepsilon$ and the expectation of the regression adjusted values, θ^* with density $\tilde q_\varepsilon(\theta,s)$ as $\tilde\theta^*_\varepsilon$. Let $E_{G,r}(\cdot)$ be the expectation with the density $G_n(v)^{(\text{norm})}$, and $E_{G,r}\{h(v)\}$ can be written as $g_{B_\delta,r}\{h(v)\}/g_{B_\delta,r}(1)$. Let $\psi(\nu)=k_n^{-1}\beta_0\{A(\theta_0)^{1/2}W_{\rm obs}+a_n\varepsilon_n\nu\}$, where $k_n=1$, if $c_\varepsilon<\infty$, and $a_n\varepsilon_n$, if $c_\varepsilon=\infty$.

Lemma 4. Assume Condition 3–6 and 7. Then if $\varepsilon_n=o(a_n^{-1/2})$,

(i) $\tilde\theta_\varepsilon=\theta_0+a_n^{-1}\beta_0A(\theta_0)^{1/2}W_{\rm obs}+\varepsilon_n\beta_0E_{G_n,r}(\nu)+r_1$, where $r_1=o_P(a_n^{-1})$;
(ii) $\tilde\theta^*_\varepsilon=\theta_0+a_n^{-1}\beta_0A(\theta_0)^{1/2}w_{\rm obs}+\varepsilon_n(\beta_0-\beta_\varepsilon)E_{G_n,r}(\nu)+r_2$, where $r_2=o_P(a_n^{-1})$.

Proof of Lemma 4: These results generalize Lemma A3(c) and Lemma A5(c) in [1] in the sense of permitting use of a data-dependent $r_n(\theta)$, however here we are only considering $\varepsilon_n=o(a_n^{-1/2})$ in contrast to Lemma A5(c) in [1] which assumes the less stringent condition that $\varepsilon_n=o(a_n^{-3/5})$.

With the transformation $t=t(\theta)$, by Lemma 2, if $\varepsilon_n=o(a_n^{-1/2})$,

$$\begin{cases}\tilde\theta_\varepsilon=\theta_0+a_{n,\varepsilon}^{-1}\tilde q_{B_\delta,t\nu}(t)/\tilde q_{B_\delta,t\nu}(1)\\ \quad=\theta_0+a_{n,\varepsilon}^{-1}g_{B_\delta,r}(t)/g_{B_\delta,r}(1)+o_p(a_n^{-1}),\\ \tilde\theta^*_\varepsilon=\theta_0+a_{n,\varepsilon}^{-1}\tilde q_{B_\delta,t\nu}(t)/\tilde q_{B_\delta,t\nu}(1)-\varepsilon_n\beta_\varepsilon\tilde q_{B_\delta,t\nu}(\nu)/\tilde q_{B_\delta,t\nu}(1)\\ \quad=\theta_0+a_{n,\varepsilon}^{-1}g_{B_\delta,r}(t)/g_{B_\delta,r}(1)-\varepsilon_n\beta_\varepsilon E_{a_n,r}(\nu)+o_p(a_n^{-1}),\end{cases}\tag{8.4}$$

where the remainder term comes from the fact that $(a_{n,\varepsilon}^{-1}+\varepsilon_n)\left\{O_p(a_{n,\varepsilon}^{-1})+O_p(a_n^2\varepsilon_n^4)\right\}=o_p(a_n^{-1})$.

First the leading term of $g_{B_\delta,r}(t\nu^k)$ is derived for $k=0$ or 1. The case of $k=1$ will be used later. Let $t'=t-\psi(\nu)$, then

$$\begin{aligned}g_{B_\delta,r}(t\nu^{k_2})&=\int_{\mathbb{R}^d}\int_{t(B_\delta)}\{t'+\psi(\nu)\}\nu^{k_2}\gamma_n(t)g_n(t,\nu)\,dtd\nu\\&=\int_{\mathbb{R}^d}\psi(\nu)\nu^{k_2}G_{n,r}(\nu)\,d\nu\\&\quad+\int_{\mathbb{R}^d}\int_{t(B_\delta)}t'\nu^{k_2}\gamma_n(t)g_n(t,\nu)\,dtd\nu.\end{aligned}$$

By matrix algebra, it is straightforward to show that $g_n(t,v)=N\{t;\psi(v),k_n^{-2}I(\theta_0)^{-1}\}G_n(v)$. Then with the transformation t', we have

$$g_{B_\delta,r}(t\nu^{k_2})-\int_{\mathbb{R}^d}\psi(\nu)\nu^{k_2}G_{n,r}(\nu)\,d\nu=$$
$$\int_{\mathbb{R}^d}\int_{t(B_\delta)-\psi(\nu)}t'\nu^{k_2}\gamma_n\{\psi(\nu)+t'\}N\left\{t';0,k_n^{-2}I(\theta_0)^{-1}\right\}G_n(\nu)\,dt'd\nu.$$

By applying the Taylor expansion on $\gamma_n\{\psi(\nu)+t'\}$, the right hand side of the above equation is equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{t(B_\delta)-\psi(\nu)}t'N\{t';0,k_n^{-2}I(\theta_0)^{-1}\}\,dt'\cdot\gamma_n\{\psi(\nu)\}\nu^{k_2}G_n(\nu)\,d\nu\\&+\int_{\mathbb{R}^d}\int_{t(B_\delta)-\psi(\nu)}t'^2D_t\gamma_n\{\psi(\nu)+e_t\}N\{t';0,k_n^{-2}I(\theta_0)^{-1}\}\,dt'\cdot\nu^{k_2}G_n(\nu)d\nu=\\&k_n^{-1}\int_{\mathbb{R}^d}\int_{Q_v}t''N\{t'';0,I(\theta_0)^{-1}\}\,dt''\cdot\gamma_n\{\psi(\nu)\}\nu^{k_2}G_n(\nu)\,d\nu\\&+k_n^{-2}\int_{\mathbb{R}^d}\int_{Q_v}t''^2D_t\gamma_n\{\psi(\nu)+e_t\}N\{t'';0,I(\theta_0)^{-1}\}\,dt''\cdot\nu^{k_2}G_n(\nu)\,d\nu,\end{aligned}\tag{8.5}$$

where $Q_v=\{a_n(\theta-\theta_0)-k_n\psi(\nu)\mid\theta\in B_\delta\}$ and $t''=k_nt'$. Since Q_v can be written as $\left\{a_n(\theta-\theta_0-\beta_0\varepsilon_n\nu)-\beta_0A(\theta_0)^{1/2}W_{\rm obs}\mid\theta\in B_\delta\right\}$, it converges to $\mathbb{R}^p$ for any fixed v with probability one using the dominated convergence theorem. Then $\int_{Q_v}t''N\{t'';0,\tau(\theta_0)^{-1}\}\,dt''=o_P(1)$ for fixed v, and by the continuous mapping theorem and Condition 4, the first term in the right hand side of (8.5) is of the order $o_p(k_n^{-1})$. The second term is bounded by

$$k_n^{-2}\sup_{t\in\mathbb{R}}\|D_t\gamma_n(t)\|\int_{\mathbb{R}^p}\|t''\|^2N\{t'';0,I(\theta_0^{-1})\}\,dt''\int_{\mathbb{R}^d}\nu^{k_2}G_n(\nu),d\nu,$$

which is of the order $O_p(k^{-2}\tau_n/a_{n,\varepsilon})$ by Condition 6. Therefore

$$g_{B_\delta,r}(t\nu^{k_2})=\int_{\mathbb{R}^d}\psi(\nu)\nu^{k_2}G_n(\nu)d\nu+o_P(k_n^{-1}).\tag{8.6}$$

By algebra, $k_n=a_{n,\varepsilon}^{-1}a_n$, and

$$\int_{\mathbb{R}^d}\psi(\nu)\nu^{k_2}G_n(\nu)d\nu=$$
$$a_{n,\varepsilon}\beta_0\{a_n^{-1}A(\theta_0)^{1/2}W_{\rm obs}\int_{\mathbb{R}^d}\nu^{k_2}G_{n,r}(\nu)\,d\nu+\varepsilon_n\int_{\mathbb{R}^d}\nu^{k_2+1}G_{n,r}(\nu)\,d\nu\}.\tag{8.7}$$

Then (i) and (ii) in the Lemma holds by plugging the expansion of $g_{B_\delta,r}(t)$ into (8.4).

□

Lemma 5. Assume Condition 3, 4, 2–9. Then as $n\to\infty$,

(i) For any $\delta<\delta_0$, $r_{B_\delta^c}(1)$ and $\tilde q_{B_\delta^c}(1)$ are $o_p(\tau_n^p)$. More specifically, they are of the order $O_p\left(\tau_n^pe^{-a_{n,\varepsilon}^{\alpha_\delta}c_\delta}\right)$ for some positive constants c_δ and α_δ depending on δ.
(ii) $q_{B_\delta}(1)=\tilde q_{B_\delta}(1)\{1+O_p(\alpha_n^{-1})\}$ and $\sup_{A\subset B_\delta}|q_A(1)-\tilde q_A(1)|/\tilde q_{B_\delta}(1)=O_p(\alpha_n^{-1})$;
(iii) if $\varepsilon_n=o(a_n^{-1/2})$, then $\tilde q_{B_\delta}(1)$ and $r_{B_\delta}(1)$ are $\Theta_P(\tau_n^pa_{n,\varepsilon}^{d-p})$, and thus $\tilde q_{\mathcal{P}_0}(1)$ and $q_{\mathcal{P}_0}(1)$ are $\Theta_P(\tau_n^pa_{n,\varepsilon}^{d-p})$;
(iv) if $\varepsilon_n=o(a_n^{-1/2})$, $\theta_\varepsilon=\tilde\theta_\varepsilon+o_p(a_{n,\varepsilon}^{-1})$. If $\varepsilon_n=o(a_n^{-3/5})$, $\theta_\varepsilon=\tilde\theta_\varepsilon+o_P(a_n^{-1})$.

Proof of Lemma 5: This generalizes Lemma 2 in the supplements of [1]. The arguments therein can be followed exactly, by Condition 4 and the fact that regarding $\pi(\theta)$, only the condition $\sup_{\theta\in\mathbb{R}^p}\pi(\theta)<\infty$ is used. □

Lemma 6. Assume Condition 3, 4, 2–9.

(i) For any $\delta<\delta_0$, $Q_\varepsilon(\theta\in B_\delta^c\mid s_{\text{obs}})$ and $\tilde{Q}_\varepsilon(\theta\in B_\delta^c\mid s_{\text{obs}})$ are $o_p(1)$;
(ii) There exists some $\delta<\delta_0$ such that

$$\sup_{A\in\mathfrak{B}^p}|Q_\varepsilon(\theta\in A\cap B_\delta\mid s_{\text{obs}})-\tilde{Q}_\varepsilon(\theta\in A\cap B_\delta\mid s_{\text{obs}})|=o_p(1);$$

(iii) $a_{n,\varepsilon}(\theta_\varepsilon-\tilde{\theta}_\varepsilon)=o_p(1)$.

Proof of Lemma 6: This lemma generalizes Lemma A3 of [1]. The proof of Lemma A3 of [1] only needs Lemma 3 and 5 from [2] to hold. The result that $q_{B_\delta^c}\{h(\theta)\}=O_p(\tau_n^p e^{-a_{n,\varepsilon}^{\alpha_\delta}c_\delta})$ for some positive constants α_δ and c_δ, which generalizes the case of $r_n(\theta)=\pi(\theta)$ in Lemma 3 of [2], holds by Condition 4, since the latter only uses the fact that $\sup_{\theta\in B_\delta^c}\pi(\theta)<\infty$. Then the arguments in the proof of Lemma 3 in [2] can be followed exactly, despite the term τ_n^p that is not included in the order of $\pi_{B_\delta^c}\{h(\theta)\}$, since $Q_\varepsilon(\theta\in A\mid s_{\text{obs}})$ is the ratio $q_A(1)/q_{\mathbb{R}^p}(1)$. Since Lemma 5 in [2] has been generalized by Lemma (3) above, the arguments of the proof of Lemma A3 in [1] can be followed exactly. □

With the above lemmas holding for $\varepsilon_n=o(a_n^{-1/2})$, lines for proving Proposition 1 in [1] can be followed exactly to finish the proof of Theorem 2.

9. PROOF OF THEOREM 3

Lemma 7. Assume Condition 3–10. If $\varepsilon_n=o_p(a_n^{-3/5})$, then $a_n\varepsilon_n(\beta_\varepsilon-\beta_0)=o(1)$.

Proof of Lemma 7: This generalizes Lemma A4 in [1] by replacing $\pi(\theta_0+a_{n,\varepsilon}^{-1}t)$ therein with $\gamma_n(t)$. By Condition 4 and the arguments in the proof of Lemma A4 in [1], it can be shown that

$$\frac{q_{\mathbb{R}^p}\{(\theta-\theta_0)^{k_1}(s-s_{\text{obs}})^{k_2}\}}{q_{\mathbb{R}^p}(1)}=a_{n,\varepsilon}^{-k_1}\varepsilon_n^{-k_2}\left\{\frac{\tilde{q}_{B_\delta,tv}(t^{k_1}\nu^{k_2})}{\tilde{q}_{B_\delta,tv}(1)}+O_p(\alpha_n^{-1})\right\}.$$

Then by Lemma 2 (iii), the right hand side of the above is equal to

$$a_{n,\varepsilon}^{-k_1}\varepsilon_n^{-k_2}\left\{\frac{g_{B_\delta,r}(t^{k_1}\nu^{k_2})}{g_{B_\delta,r}(1)}+O_p(a_{n,\varepsilon}^{-1})+O_p(a_n^2\varepsilon_n^4)+O_p(\alpha_n^{-1})\right\}.$$

Since $\beta_\varepsilon=\text{Cov}_\varepsilon(\theta,S_n)\text{Var}_\varepsilon(S_n)^{-1}$,

$$a_n\varepsilon_n(\beta_\varepsilon-\beta_0)=k_n\left[\frac{g_{B_\delta,r}(t\nu)}{g_{B_\delta,r}(1)}-\frac{g_{B_\delta,r}(t)g_{B_\delta,r}(\nu)}{g_{B_\delta,r}(1)^2}+o_p(k_n^{-1})\right]\times$$
$$\left[\frac{g_{B_\delta,r}(\nu\nu^T)}{g_{B_\delta,r}(1)}-\frac{g_{B_\delta,r}(\nu)g_{B_\delta,r}(\nu)^T}{g_{B_\delta,r}(1)^2}+o_p(k_n^{-1})\right]^{-1}-a_n\varepsilon_n\beta_0,$$

where the equations that $a_{n,\varepsilon}^{-1}k_n=o(1)$, $a_n^2\varepsilon_n^4k_n=o(p)$, and $\alpha_n^{-1}k_n=o(a_n^{-2/5}k_n)=o(1)$ are used. By algebra, the right hand side of the equation above can be rewritten as

$$\left\{\frac{g_{B_\delta,r}\{(k_nt-a_n\varepsilon_n\beta_0\nu)\nu\}}{g_{B_\delta,r}(1)}-\frac{g_{B_\delta,r}(k_nt-a_n\varepsilon_n\beta_0\nu)g_{B_\delta,r}(\nu)}{g_{B_\delta,r}(1)^2}+o_p(1)\right\}\times\left\{E_{G,r}(\nu\nu^T)-E_{G,r}(\nu)E_{G,r}(\nu)^T+o_p(k_n^{-1})\right\}^{-1}.$$

By plugging (8.6) and (8.7) in the above, $a_n\varepsilon_n(\beta_\varepsilon-\beta_0)$ is equal to

$$\begin{aligned}&\{E_{G,r}(\nu)\beta_0A(\theta_0)^{1/2}W_{\text{obs}}-E_{G,r}(\nu)\beta_0A(\theta_0)^{1/2}W_{\text{obs}}+o_p(1)\}\times\\&\{\text{Var}_{G,r}(\nu)+o_p(k_n^{-1})\}^{-1}\\&=o_P(1)\{\text{Var}_{G,r}(\nu)+o_p(k_n^{-1})\}^{-1}.\end{aligned}$$

Since

$$\text{Var}_{G,r}(\nu)\geq\frac{\inf_{t\in t(B_{\delta'})}\gamma_n(t)}{g_{B_\delta,r}(1)}\int_{\mathbb{R}^d}\int_{t(B_{\delta'})}\{\nu-E_{G,r}(\nu)\}^2g_n(t,\nu)\,dtd\nu,$$

where δ' is defined in the proof of Lemma 3(ii), we have $\text{Var}_{G,r}(\nu)^{-1}=\Theta_p(1)$. Therefore $a_n\varepsilon_n(\beta_\varepsilon-\beta_0)=o_p(1)$. □

Lemma 8. Results generalizing Lemma A5 in [1], *i.e.* replacing Π_ε and $\tilde{\Pi}_\varepsilon$ therein with Q_ε and $\tilde{Q}_\varepsilon$, hold.

Proof of Lemma 8: In [1], the proof of Lemma A5 in [1] requires Lemma A4 and Lemma 2 in the supplements therein to hold. Since we have show that their generalized results hold for $\varepsilon_n=o_p(a_n^{-1/2})$, (see Lemma 7 and Lemma 5 above), the proof of this lemma for $\varepsilon_n=o_p(a_n^{-3/5})$ follows the same arguments in [1], replacing $\pi(\theta)$ with a $r_n(\theta)$ that satisfies conditions 3 – 6. □

With all above lemmas, the proof of Theorem 3 holds by following the same arguments as those in the proof of Theorem 1 in [1].

REFERENCES

[1] Li, W. and Fearnhead, P. (2018). Convergence of regression-adjusted approximate Bayesian computation. *Biometrika* **105**(2) 301–318.

[2] Li, W. and Fearnhead, P. (2018). On the Asymptotic Efficiency of approximate Bayesian computation estimators. *Biometrika* **105**(2) 286–299.